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GEOMETER'S ANGLE

Islamic Geometric Patterns in Higher Dimensions

Sam Moradzadeh[1] · **Ahad Nejad Ebrahimi**[1]

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Abstract

The purpose of this paper is to develop the Islamic geometric patterns from planar coordinates to three or higher dimensions through their repeat units. We use historical plane methods, polygons in contact (PIC) and point-joined, in our deductive approaches. The mentioned approach makes use of a novel method of tessellation that generates 3D Islamic patterns called "interior polyhedral stellations". The outputs showed that both the PIC and point-joined methods have strengths and weaknesses. Point-joined stellations are more efficient for regular repeat units and PIC is suitable for complex designs. These two methods can produce a large range of patterns and can be employed simultaneously. This study effectively answers the question regarding the gap between planar design from Muslim achievements and contemporary demands in modern art and architecture. We also propose techniques for constructing aperiodic three-dimension Islamic geometric patterns tessellation and two-point family.

Keywords Islamic ornaments · Geometric algorithm · Space tessellation · Dimensions · Symmetry

Introduction

The studies of Islamic arts and design have shown that Muslim artists had a vision for developing their design consistency. They developed their design from simple cross-star (grid) patterns to complex dual-level designs. In Fig. 1b, there is an example of an Islamic pattern projected into an octahedron, one of the Archimedean solids. This

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✉ Sam Moradzadeh
sam.moradzade@gmail.com; s.moradzadeh@tabriziau.ac.ir

Ahad Nejad Ebrahimi
ahadebrahimi@tabriziau.ac.ir

[1] Department of Architecture and Urbanism, Tabriz Islamic Art University, Azadi Blvd, Hakim Nizami Ganjavi Sq, Tabriz, Iran

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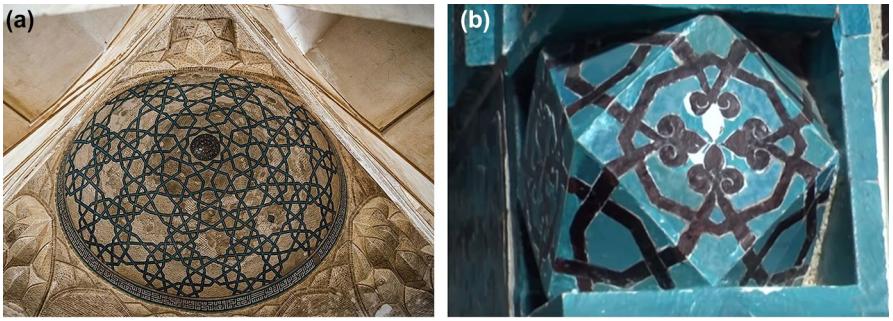


Fig. 1 Historical developments of planar pattern; **a** Islamic pattern projection on the inner dome of Jameh Mosque of Saveh, Iran, Photo: Tasnim News Agency, licensed under the CC BY 4.0 **b** Projected obtuse pattern on octahedron in the Mehrab of Eşrefoğlu Mosque, Konya, Turkey, Photo: Arash Mohamadi, reproduced by kind permission

example in the Eşrefoğlu Mosque demonstrates knowledge about geometric solids and geometrical projection in the thirteenth century. The other example of plane pattern development is the geometric projection on domes shown in Fig. 1a. Bonner (2017b) introduced two historical repetitive stratagems for applying geometric designs onto the surfaces of domical structures: polyhedral symmetry and radial gore segments. It is essential to mention here that existing historical and traditional examples are either planar patterns or the projection of a three-dimensional plane on polyhedra or curve slices. Therefore, the output in both methods is the surface. The surface has various definitions regarding the field of study as Peter Firby and Cyril Gardiner described: "the space models of many of the surfaces which concern us sit naturally in 3 (or higher)-dimensional space. But soon we shall meet surfaces that do not sit usually in 3-dimensional space. They appear to be unduly restricted and cry out to be represented in a higher-dimensional space" (2001: 18). Simply stated, a surface is basically a 2D face and identifies an area, whilst the part that surrounds it is 3D and has a volume. For example, the surface of a vertical plane or sphere divides the region of one side from another or the interior from the exterior.

Islamic art and ornament feature muqarnas, which have a unique structure and design. Muqarnas, or stalactite vaulting, is a three-dimensional ornament that extrudes a horizontal layer of patterns and connects them with vaults (Fig. 2). Bloom (1987) describes them as an ornament that gives the ability to distinguish between the main parts of a building and serves as a transition from the walls of a room into a domed ceiling. Muqarnas is a system of projecting niches used for zones of transition and architectural decoration (Necipoğlu and Leal 2010).

It seems muqarnas lie in the pathway of planar expansion to higher space. Even though in plan muqarnas share some similar motifs with girih (for example, star motifs), Islamic scholars do not group them as the 3D patterns. Other structures in Islamic architecture such as karbandi and kase-sazi (Mohammadi et al. 2018), an intersection of several vaults, that have similar characteristics are also not considered to be three-dimensional patterns.

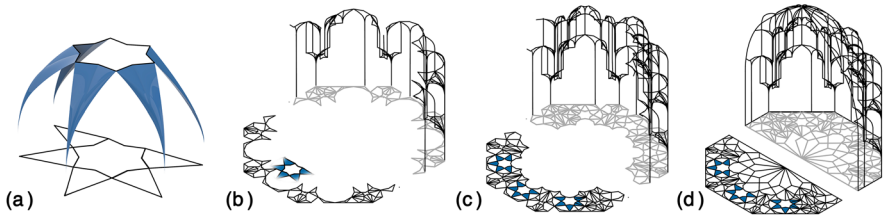


Fig. 2 The four steps of creating *Muqarnas* vault from plan: **a** specifying components; **b** defining the levels of vaults; **c** creating vaults; **d** primary vaults in 3D view

After this introduction, the question that we face is: that would muqarnas be a 3D Islamic geometric pattern (girih) with our perspective? Before answering the question, let us first review the rules of Islamic geometric tessellation:

1. They must tessellate the plane infinitely.
2. Repeat units must not create overlaps or gaps in tiling.

By comparing the rules for muqarnas with those for girih, we can conclude that muqarnas is not a 3D girih. Muqarnas cannot tessellate the space entirely and historically their usage has always been different. There is a controversy among the muqarnas and girih scholars regarding the origin of each ornament. Some believe that the muqarnas form was derived from girih, while others believe that muqarnas developed independently. For these reasons, we are looking to fill the gap between 2D patterns and 3D ornament. We are concentrating on plane patterns and transforming them into 3D separately thanks to the use of mathematics and geometry. This paper aims to develop 2D designs for contemporary usage in both architecture and art. In our contemporary era and with our tools, what would our answer be? How can we employ this new type of form in our architectural space? We are trying to answer this kind of question.

A few comments on the subject and terminology. Geometric design in Farsi has the specific word, *girih*, meaning "knot"; therefore, we have chosen to employ the phrase "Islamic geometric pattern" for the sake of clarity of scientific content. We use "repeat unit" as a generic term for a template that is repeated using three types of symmetries: rotational, reflection and point symmetry. For other tiling and substitution tiling phrases, we refer to the terminology in Grünbaum and Shephard (1987).

Related Works

In this section, we survey the researches and their approaches that expanded Islamic geometric patterns, both mathematically and geometrically. After visiting the Alhambra M. C. Escher had a great impact by taking the idea of abstract patterns and ornaments. He did not use the Islamic patterns directly, but took the

symmetry and arrangement of the patterns and applied them in his works of art. "The regular division of a plane is the richest source of inspiration that I ever struck; nor has it yet dried up" (Escher 2000). He also described the symmetry operations on the figures and following pages show how a surface can be regularly divided into or filled with similar shaped figures which are contiguous to one another, without leaving any gaps (2000; Schattschneider 1990).

Dunham (2001) extended the process by exhibiting repeating Islamic patterns in hyperbolic geometry, the third classical geometry. He has advanced a theoretical framework that makes it possible to create hyperbolic Islamic patterns that are related to Euclidean patterns. At the end of his research, Dunham suggested creating a hyperbolic version of other families of Islamic patterns. He also suggested a method consisting of his and Kaplan's (2000) algorithm to generate hyperbolic star patterns.

A more recent example is the work of Craig Kaplan, who successfully applied computer science and wrote an applet for developing Islamic patterns (Kaplan 2003). He also designed Islamic patterns on a non-Euclidian sphere and even on the Stanford Bunny. His works have presented a construction technique for creating a broad set of traditional Islamic star patterns, along with interesting original designs. They have also shown how these patterns can be constructed in a way that is independent of Euclid's parallel postulate, allowing them to be adapted to the sphere or the hyperbolic plane in addition to the Euclidean plane (Kaplan 2005; Kaplan and Salesin 2004; Bonner and Kaplan 2017).

Riether and Baerlecken (2012) use Grasshopper for scripting an algorithm based on the polygon in contact (PIC) method for manipulating the girih surface. Their Digital Girih project was built based on the research of Lu and Steinhardt (2007) and their findings on aperiodic patterns in medieval Islamic architecture. The Girih Project is a quasi-crystalline system that intensifies the relation of pattern and volume. The volume in their project emerges from changing matching rules within a pattern. In that way, it is a bottom-up process that depends on the matching rules used for the pattern generation. However, they have not achieved the exact 3D girih since they retouched the plane patterns and produced a surface. As we mentioned earlier, surfaces have area, but they can create a form, so they called their algorithm 2.5-dimensional pattern generator.

Jay Bonner has done many excellent works. In his latest book, *Islamic Geometric Patterns* (Bonner 2017b) he also contributed several methods for mathematically expanding upon the traditional practices and design types associated with Islamic geometric patterns. These include fivefold aperiodic patterns that employ Sir Roger Penrose's matching rules; sevenfold patterns with scale-invariant self-similarity; and recursive quasi-periodic fivefold patterns with diminishing scale. In separate research, he used Buckminster Fuller's Jitterbug method for projecting Islamic patterns, and the result is very similar to the historical polyhedron projection in Konya, Turkey (see Fig. 1b). These 3D jitterbug transformations have 2D corollaries that are instructive in understanding this design process. As he writes, "this expands upon the historical use among some Muslim cultures of polyhedral geometry as an organizing principle for placing geometric patterns onto the surfaces of domes and domical niches and

provides contemporary artists with a new approach to applying the geometric design onto spherical surfaces" (Bonner 2018).

One article very close to our approach is that of Reitebuch et al. (2018). They created shapes based on the construction of 2D girih patterns, which provided a corresponding procedure to construct 3D girih patterns covering $\mathbb{R}3$ completely. They have achieved five different tilings of space: the first consists of octagon prisms and truncated cuboctahedra, the second contains truncated octahedra, the third uses cubes, truncated cuboctahedra, and truncated octahedra, the fourth is built of octagon prisms and cubes, and the fifth is made of cubes only. We elaborate and improve upon their (Reitebuch et al. 2018) work in several ways. First, we provide two systematic methods—interior polyhedral stellations and point-joined—to generate 3D motifs and more useful parameterizations for Islamic geometric patterns. Second, these algorithms can tessellate through periodic and aperiodic space, also in higher dimensions. Finally, and most significantly, we suggest and discuss the application of these solids and shapes in contemporary art and architecture.

Mathematical Background

There is an agreement among scholars about the great mathematics behind medieval Islamic pattern and ornaments. Jan Hogendijk (2012) clarifies the relation between mathematician-astronomers and craftsmen-artist, concluding that mathematician-astronomers worked with geometric proofs in the style of Euclid's *Elements*. Craftsmen were familiar with the Euclidean way of drawing figures, using letters as labels of points. However, he believes that craftsmen did not use geometric proofs and were not trained in the methods of Euclid's *Elements*. Abu'l-Wafa participated in solving the problems in artisans and craftsmen's designs. Alpay Özdural says that he seems to have enjoyed being involved with architecture and architectural arts, the two rewarding fields for the applications of geometry (Özdural 2000; Necipoğlu 2017). Thus we can emphasize the importance of mathematics in medieval designs and arts. For this reason, in this section, we will review the required applied mathematics materials that we will use for transforming Islamic geometric patterns.

To present a comprehensive method to expand the idea, we provide a technical introduction to the modern mathematical concepts and materials in this part. More information and technical details about the relations between Islamic geometric patterns, mathematics, and geometry can be found in some of the excellent texts (Abas and Salman 1995, p. 232; Wichmann and Wade 2017; Thomas and Hann 2007; Horne 2000; Bonner 2017a).

As we mentioned earlier, there are two fundamental rules for Islamic geometric patterns (*girih*). First, the repeat units have a specified ornament that is defined under separate families (acute, media, obtuse, and two-point) (Bonner 2017b). Second, repeat units can tessellate infinitely with the condition of no overlaps and no gaps. By knowing these rules, we can determine the mathematics required for geometric transformation. For adding higher dimensions to the planar repeat unit, we will describe dimensions in mathematics first. Due to the comprehensiveness of

the PIC method, we have chosen this specific technique as the primary method. For this reason, we will explain polyhedra and polygons in the next section. And finally, because of the second rule, we will study the types of tessellations and tilings.

Dimensions

The term "dimension" generally defines a measure, parameter or properties of an object, for example, length, width, height, and depth. In classical physics, for a point in space, three physical dimensions are considered, the fundamental aspects of this structure being expressed in terms of forward and backward, right, left and up and down. However, in differential geometry dimension can take a different meaning. In Fig. 3, we show an example of changes in dimensions.

In mathematics, the dimension of an object represents the number of degrees of freedom of a point that shifts on that object. In simple words, the number of dimensions defines independent parameters or coordinates that are needed for determining the location of that specified coordinates on an object. For example, a point has zero dimensions because a point can only have one independent parameter; a line is one dimensional due to the movement of a point on the line in just one direction. Sommerville (1958) describes n dimensions as a topological space, meaning that there is possibly a higher dimension in mathematics, although the human cannot process them. For example, in the superstring theory, eleven dimensions are required for the base equation. Below we will exam the possibility of Islamic geometric pattern transformation into 4D space.

The 2D design in 3D space considers a plane; the equation for that is Eq. (1) and the normal of the plane is defined by $\overrightarrow{N} = (a, b, c)$. A random point on the plane has the coordinate of $A(x_1, y_1, z_1)$ and for our purpose, we should project the point to space by manipulating the Z-axis. This new point, by having z value, can represent the three dimensions. Now by solving the Eq. (1) we can project every planar point into space, and finally, a solid will form in the space.

$$ax + by + cz + d = 0 \qquad \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{Z - z_1}{c} = k \tag{1}$$

As we described earlier, by going to higher dimensions, there is a higher degree of freedom, meaning a different angular situation in the objects. In the PIC method, the plane angle (α) on a segment line can have variables of 0° to 180°. Technically these two angles are impossible as inputs because the result equals

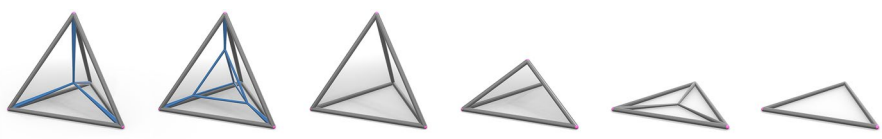


Fig. 3 Geometric dimension transformation from 2D into 4D; an equilateral triangle, into tetrahedron and tetrahedron into regular 5-cell or pentachoron

zero in the equation. In three dimensions, we have a solid angle instead of the plane angle. Solid angle (Ω) measures how large an object is and makes a cone of view from a particular viewing point. The unit for the solid angle is steradian or square radian (sr).

$$\alpha = 2 \times \arccos\left(1 - \frac{\Omega}{(2\pi)}\right) \quad \Omega = 2\pi \times \left(1 - cos\left(\frac{\alpha}{2}\right)\right) \tag{2}$$

Equation (2) is the planar and space converter tool for transmuting our four plane design families (acute, median, obtuse and two-point) into space angle.

Polyhedra and Polygons

Polygons are 2D (plane) figures that are built from line segments. Polyhedra are 3D solids which consist of polygons forming faces, edges, and vertices. A polyhedron is not the exact 3D polygon, although sometimes polyhedra are created by regular polygon faces, like the cube created by squares and the tetrahedron created by equilateral triangles. We can operate geometric transformations of polygons to polyhedra and polygons to polygons. In operating from a higher dimension to a lower one, there is a partial loss of symmetry. For example, in the transformation of a cube into a network square, depth or edges are projected into lower dimensions.

Truncation is one of the operations that cut polyhedron vertices; the result is the new face in place of each vertex (Fig. 4). The actual vertex figures of a regular polygon (p) are the sides of another (p) which we may call a truncation of the first (Coxeter 1973). This operation can be used as a secondary method for 3D Islamic geometric patterns after they are transformed from 2D.

Leonard Euler discovered the formula $V - E + F = 2$ describing the number of vertices, edges, and faces of a convex polyhedron (Cromwell 1999). This formula works with all of the polyhedra except those with a hole running through them, which are called non-simple polyhedra. Euler's formula discloses the fact that there is no simple polyhedron that has ten faces and seventeen vertices. The real usage of this formula in our method is the evaluation of the polyhedra after transformation into 3D. If the number of each relative variable is correct, that means we have the right result.

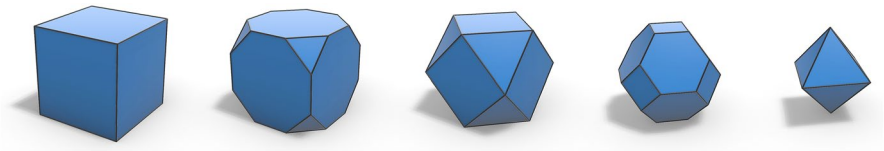


Fig. 4 Progressive truncation of the cube into truncated cube, cuboctahedron, truncated octahedron, and octahedron

Tessellation and Tiling

If we tile a plane using one or more geometric shapes (polygons) without gaps or overlaps, it is called tessellation. Different geometric shapes (the so-called repeat unit) result in various kinds of tessellation. An *n-uniform* tessellation is an edge-to-edge configuration of tiling of regular polygon faces with the condition of distinct transitive vertices that *n* defines the distinctively.

Regular convex polygons that have equal sides and inside angles produce regular tessellations which are squares, equilateral triangles and hexagons. Semi-regular or Archimedean tessellations are the other convex polygons that have all vertices belonging to the same transitivity class. Still, their repeat units are a combination of regular shapes that make eight possibilities for this type of tiling. Each vertex or intersection point has the same pattern of geometric shapes. As Bonner (2017b) described, regular, semi-regular, two-uniform, and three-uniform tilings were used historically as underlying tessellations to generate Islamic geometric patterns.

Tiling the space or 3D close packing tessellation, also called honeycomb, is mathematically possible with polyhedra (Coxeter 1973). However, there is no unanimous agreement about the number of conditional possibilities with polyhedra or polytopes that tessellate the space. In Table 1, we have summarized some of the mathematicians' theories about space-filling tessellation.

As with *n-uniform* plane tessellation, there is *n-uniform* honeycomb in which *n* defines the number of dimensions. In our approach, we are working with *three-uniform* honeycombs which consist of uniform polyhedron units having the same vertices groups. In three dimensions, there is just one regular honeycomb, which has eight cubes at each polyhedron vertex. Similarly, in three dimensions, there is only

Table 1 Space filling tessellation summarization

Scholar	Description
(Gardner 1971, pp. 183–184)	The *cube* is the only Platonic solid possessing this property
(Steinhaus 1999, p. 210; Wells 1991, p. 232)	The *cube* and a combination of *tetrahedra* and *octahedra* do fill space, *The rhombic dodecahedron, elongated dodecahedron, trapezo-rhombic dodecahedron*
(Wells 1991, p. 235)	In addition, *octahedra*, *truncated octahedron*, and *cubes*, combined in the ratio 1:1:3, can also fill space
(Wells 1991,p. 234)	In 1914, Föppl discovered a space-filling compound of *tetrahedra* and *truncated tetrahedra*
(Steinhaus 1999, pp. 185–190; Wells 1991, pp. 233–234)	There are only five space-filling convex polyhedra with regular faces: *the triangular prism*, *hexagonal prism*, *cube*, *truncated octahedron*
(Johnson 1966)	*Gyrobifastigium*, *the triangular prism*, *hexagonal prism*, *cube*, *truncated octahedron*
(Coxeter 1973, p. 29)	*The cube, hexagonal prism, rhombic dodecahedron, elongated dodecahedron, and truncated octahedron*
(Grünbaum 1994)	*Cube (*the only Platonic polyhedron to do so*), the rhombic dodecahedron, the truncated octahedron, triangular, quadrilateral,* and *hexagonal prisms*

one quasiregular honeycomb, which has eight tetrahedra and six octahedra at each polyhedron vertex (Sherk et al. 1995). Similar to Archimedean tiling, two or more different polyhedrons can combine to tessellate the space. We are calling them semi-regular space-filling polyhedra due to their characteristics.

Approach

This paper answers the questions with two approach methods. In the deductive approaches, we extend the results by going from theories to data. In our cases, theories will be the construction methods in planar or in general Islamic geometric pattern methodologies, which Bonner has described in detail (2017b) and which were historically employed to generate patterns. In our 3D projection, the theories will be the same methodologies in the past transformed into higher dimensions, and data are original patterns due to the absence of such ornaments in the past. In the theory of Islamic geometric patterns, PIC is a comprehensive method that has a broad range of applications, from simple patterns to complex dual-level designs. For technical details, analyzed history and methodology, we recommend (Bonner 2003, 2017b; Makovicky 1992; Cromwell 2009; Hankin 1925). The PIC and point-joined methods are the two main methods in our deductive approach that employs historical theories in order to produce data (Fig. 5).

However, in the inductive approach, we are going from data to theories. The data, in this case, will be the same as in the deductive approach, but the studies on them are theoretically different. In our inductive approach, we demonstrate original data that does not have the same planar pattern, although it has the same property as 2D patterns. In the second approach, we are not transforming patterns from 2 to 3D, but ornaments forming from polyhedra directly. Therefore, the results for the method and data will be both original and theories in this approach are no longer historical. The operation of truncation in geometry is an example of a method that does not have a historical basis and can be employed for creating Islamic geometric pattern forms.

In Islamic geometric patterns, colors and graphics are added after the completion of the design. As a result, designs are the combination of intersected lines that color, define the area and induct a surface among the lines, which Castera (2016) called positive and negative spaces. In three dimensions, we can truly form a space among the intersected lines. This feature can create more possibilities for creating form.

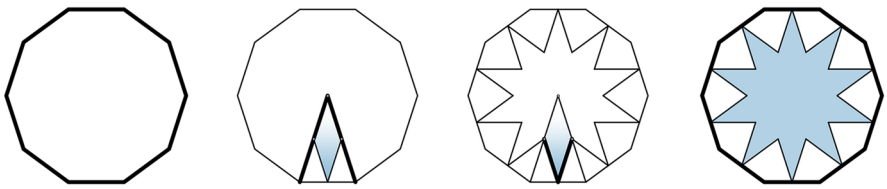


Fig. 5 Ten-pointed star construction using decagon bases and PIC method as one of the theories in deductive approaches

For example, in Fig. 6, we show how two different surfaces with the same structure can be created. We have chosen to select the simplest intersection, which generates a less complex design. In both approaches, all the surfaces on each part meet on a point, not an edge.

Construction Method

Polyhedra in Contact

In this section, we use the mathematics that were introduced in the previous section for transforming geometric patterns from the plane to a higher dimension. We show a step-by-step method for transformation by an example and describe every stage. The process for every repeat unit will be the same. In Table 2, we have chosen a square as the repeat unit. In Step 1, in the center of the repeat unit in the square, there are four line segments that surround the plane and create a polygon. This centroid point, in 3D, should have the same characteristic. In the cube, we have four faces in each direction that surrounds the space. This feature can assist us with a complex transformation from 2 to 3D.

Steps 2 and 3 are the transition of intersected lines. The points from the center of lines converge to the center of faces in 3D. Also, angles change from planar to spatial or degree to steradian, described before. Although there is a range of 0° to 180° for angles, we have selected conventional angles that exist in medieval Islamic designs: acute, median and obtuse. The fourth family, two-point, will be described separately below.

Step 4 is the intersection of cross lines from the centroid faces that create the final shape. These lines form the solid edges. Further, to complete the shape, the faces will be drawn. These faces have the same roles as the color had on the plane: they separate the space for a better understanding of the designs.

Point-Joined-Grid

Since the 1970s, the point-joining technique has been advanced by a number of proponents, causing it to gain support as the dominant historical design methodology (Bonner 2017b). There is no comprehensive definition of the point-joined method;

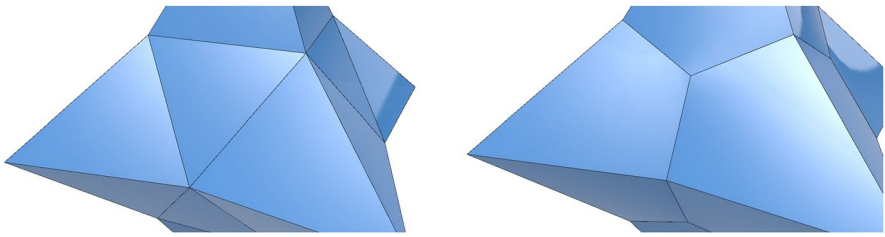
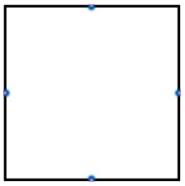
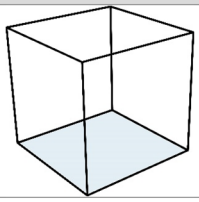
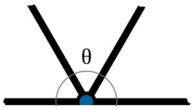
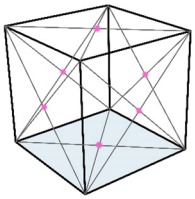
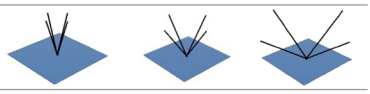
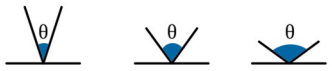
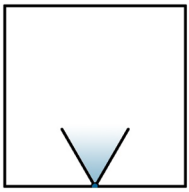
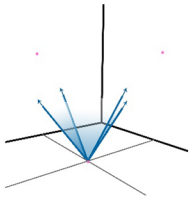
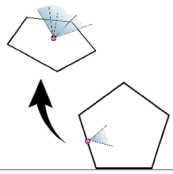
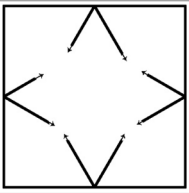
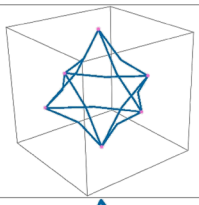
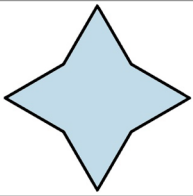


Fig. 6 Two possibilities for the intersection of faces with the same four-pointed star pattern

Table 2 Step-by-step operation for geometry transformation from 2 to 3D Islamic geometric patterns

Stages	Two dimensions	Three dimensions	Operation
Step 1			Transformation geometry matrix $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\\z\end{bmatrix}\sim\begin{bmatrix}\frac{x}{z}\\\frac{y}{z}\\1\\1\end{bmatrix}$
Step 2			Acute Median Obtuse   $\theta = 36^\circ$ $\theta = 72^\circ$ $\theta = 108^\circ$
Step 3			Angular Transformation in the centroid 
Step 4			Intersected lines from thr centroid of faces will meet each other and create a wireframe solid
Step 5			Intersected surfaces from the line boundaries complete the geometry shape and create the final polyhedron or 3D Islamic geometric patterns

as Bonner describes it, this method involves the use of a compass and straight edge. Every scholar developed the method in their own style, and there are differences in every technique. Broug (2008, 2013) is among those who expanded the point-joined method and demonstrated the process in his books. What's more, he manages the School of Islamic Geometric Design, where his ideas and methodology are



Fig. 7 Eight-pointed star pattern with the point-joint technique, described as fourfold patterns in Broug (2013, 2008)

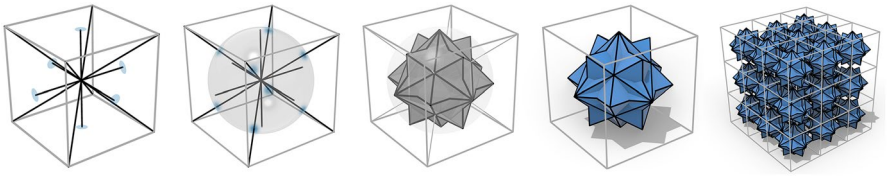


Fig. 8 Transformation of the same pattern in Fig. 7 into 3D, using spatial point-joined method

developed. In Fig. 7, we demonstrate a common eight-pointed star pattern that starts with a circle in a square, divided into eight equal sections.

First, two squares are drawn inside the circle. In later steps, the design will be completed by tracing parts of the lines in previous steps. The greatest difference between the point-joined and PIC methods is that the point-joined method employs circles, arches, and rays. To project this method from the plane to three dimensions, we are transforming the same item from 2D to the next dimension. In the early stages, the circle transforms into the sphere and square to the cube. After preparing the work template, two squares transform to the eight cubes that rotate on the centroid of the main cube (repeat unit) (Fig. 8).

As Bonner (2017b) described, in addition to the polygonal technique, many of the less complex Islamic geometric patterns can also be produced from the point-jointed method. We have presented this method to show the possibility of a different method for creating 3D Islamic geometric patterns as a deductive approach. Although this method has an excellent ability for producing patterns, PIC is a more technical method for our goals, and the transformation of polygons into 3D is less complicated than curved surfaces like spheres. For this reason, we are producing and analyzing the results based on PIC as the primary method.

Results

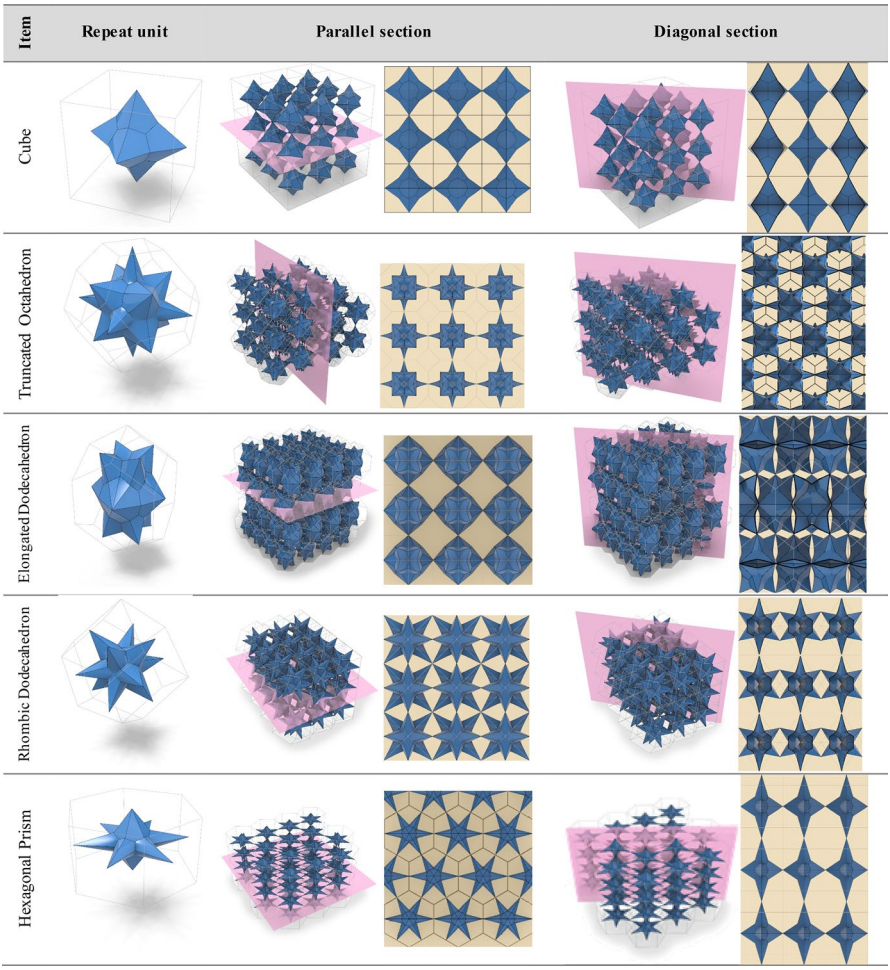
In this section, we will gather the results of our proposed method, interior polyhedral stellations, for generating 3D Islamic patterns. We categorize the results based on the symmetries of the repeat unit: *regular*, *semi-regular*, and *irregular*, all of which tessellate with *three-uniform* honeycomb in Euclidean space. The *two-point* type is a specific repeat unit that is produced with the PIC method. The results are summarized using both the inductive and deductive approaches. This method

can apply to any *n-uniform* honeycomb polyhedra and, with that said, there are various polyhedra for generating 3D patterns. It is interesting to note that historical *regular*, *semi-regular*, *two-uniform*, and *three-uniform* tessellation patterns, after transformation to the 3D shapes, can create both the same and an original design depending on their viewpoints (Table 3).

Regular

According to Table 1 and based on the conclusion of geometrical mathematicians, we have considered cube, truncated octahedron, elongated dodecahedron, rhombic dodecahedron, and hexagonal prism as the polyhedra that fill space periodically.

Table 3 Tessellation of cube, truncated octahedron, elongated dodecahedron, rhombic dodecahedron, and hexagonal prism to produce Islamic geometric patterns repeat unit in 3D



Therefore, by using the steps shown in Table 2 for these solids, Table 3 shows the 3D Islamic geometric patterns that result from regular repeat units.

Semi-regular

This family is the combination of two or more regular repeat units and is named based on the Archimedean polyhedra, semi-regular shapes. Regular shapes can produce two plane patterns, parallel and diagonal sections. This category, however, can generate more 2D plane patterns due to the more complexity in repeat units and tessellation.

Irregular

Irregular repeat units are cells that have unequal angles and unequal sides, as opposed to regular repeat units. Irregular polygons mostly fill the space between regular unit cells in Islamic geometric pattern design. The resulting group has a more historical characteristic, due to the combination of regular and irregular unit cells which was similar in traditional patterns (Table 5).

Two-Point Type

Bonner (2003) introduced a new family alongside the acute, obtuse and median angles for generating patterns with the PIC method. This type of design is essentially based on the polygonal methodology rather than a historically-based method (Fig. 9). Two-point patterns have been seen and applied in the past; however, there are no historical documents or scrolls that show the two-point underlays of the patterns. Also, these types of patterns in other methodologies (for example point-joined or grid methods) have no specific classifications and are mainly sub-categorized within the three mentioned families. Nevertheless, due to our having selected PIC as the main technique for this present article, we are projecting a two-point plane design to the three dimensions likewise.

The name of the two-point comes from the two parameters on each segment line of the polygons. Therefore, according to our interior polyhedral stellations, we have four parameters (points) on every face. These extra variables on faces create more

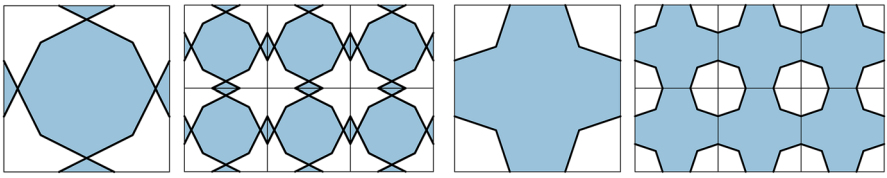


Fig. 9 Two-point pattern repeat unit and their tessellations: two-point with reflected 22.5° angles (left), two-point with 60° angles (right)

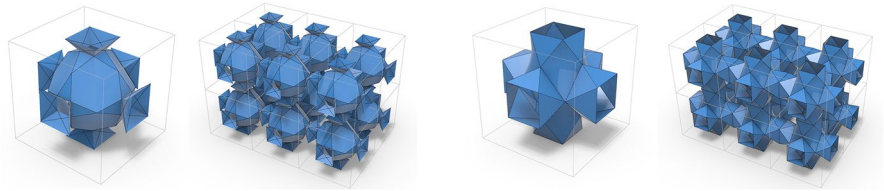


Fig. 10 The pattern in Fig. 9 transformed: using PIC method and tessellated through cube repeat unit

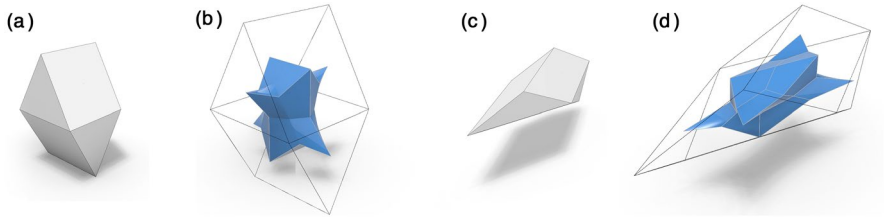


Fig. 11 **a** The gyrobifastigium; **b** Islamic geometric patterns pattern with gyrobifastigium repeat unit; **c** Schmitt–Conway–Danzer biprism; **d** Islamic geometric patterns with Schmitt–Conway–Danzer biprism repeat unit

engaging parameters in unit cells in comparison with other types, in which only one point connects the repeat units.

In Fig. 10, we show the results of the new patterns. This family has a different composition with other repeat units for creating motives. Because of the four variables on each face, there is no situation for the creation of one-point faces as a structure.

Aperiodic

In periodic tilling, repeat units tessellate through general symmetry groups. In a plane, there are seventeen repetitive patterns composing a symmetry group called the Wallpaper Group. Syed Jan Abas and Amer Shaker Salman (1995) reviewed the wallpaper group that was employed in Islamic geometric patterns in detail. Covering a surface with 5, 8, 10, and 12 axes of symmetry with only translational symmetry is impossible and result in gaps in the tiling. One of the most famous aperiodic tilings is the Penrose tiling. Penrose tilings feature a finite number of shapes known as prototiles, which can tile the plane in fulfilment of the rules of no gaps and no overlaps (Gardner 1977).

Whereas our goal is to produce 3D Islamic geometric patterns, the problem in this section is that a Penrose tiling is planar and has 2D characteristics. We are looking for polyhedra that can tessellate and fill the space aperiodically without any transitional symmetry. The *gyrobifastigium* (Fig. 11a) is a solid consisting of two conjoined triangular prisms that can fill space. In 1993 J. H. Conway introduced a polyhedron that has aperiodic and later, through a collaboration with

others, called it Schmitt–Conway–Danzer biprism (Fig. 11c) (Danzer 1995). The Schmitt–Conway–Danzer biprism is a polyhedron topologically equivalent to the gyrobifastigium, but with parallelogram and irregular triangle faces instead of squares and equilateral triangles. Like the gyrobifastigium, it can fill space, but only aperiodically or with a screw symmetry, not with a full 3D group of symmetries (Senechal 1996).

As we described in the inductive approach, by applying the same PIC method in polyhedron and without historical samples, we can achieve aperiodic 3D Islamic geometric patterns (Fig. 11b, d). Because Schmitt–Conway–Danzer biprism is the gyrobifastigium deformation solid, we can parametrize patterns by shifting from one shape to another. This script can create hybrid periodic-aperiodic Islamic geometric patterns and in a variable situation (Fig. 12).

Discussion

In this section, we discuss the application of the results obtained in the previous section. According to our goals, the results can develop along a certain path in order to employ them in architecture and art. Although some of these operations have been used before in plane design and by other researchers, we also have developed their ideas here.

Parquet Deformation

Parquet, generally speaking, is a type of floor covering, typically of wooden tiles of different shapes. William Huff, an American architecture professor, used the term "parquet deformation" in the 1960s and later Douglas Hofstadter developed this idea (Hofstadter 2008). This method is a transformation in regular tile patterns that keeps the regularity of tilings and simultaneously changes form. Huff was mostly interested in the method of attraction that deforms the pattern from one side to another. M. C. Escher used this technique with abstract and original patterns in his *Metamorphosis* collections. Craig Kaplan was among the first to connect this technique with Islamic

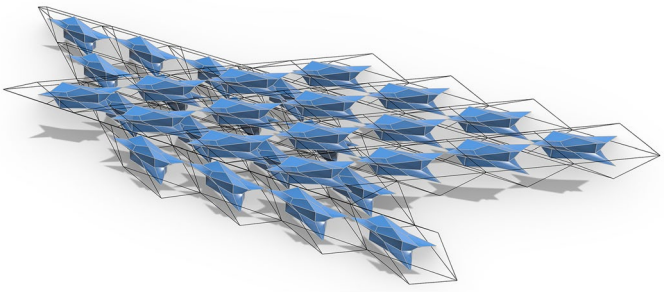


Fig. 12 The Islamic geometric patterns with Schmitt–Conway–Danzer biprism repeat unit tessellation, results in aperiodic tiling

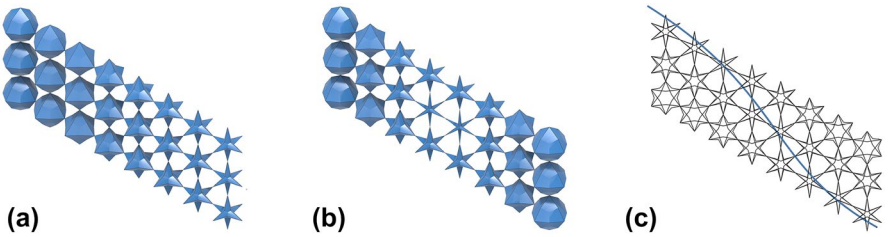


Fig. 13 Modified parquet deformation; attraction method: **a** side attracting; **b** middle attraction point; **c** attraction curve

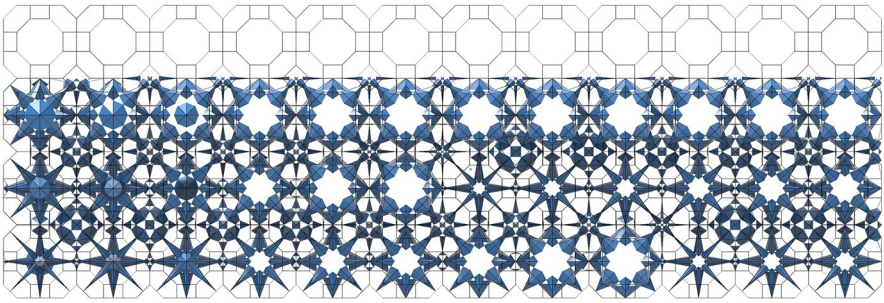


Fig. 14 3D Islamic geometric patterns parquet deformation in elevation view with unit cells

geometric patterns. He also generated Islamic geometric patterns in non-Euclidian space (Kaplan 2005).

The principles in parquet deformation are one-directional transformation and regularity of tiles. Due to our method, which develops the design in three dimensions, we can modify the rule in parquet deformation by adding dimensions in the direction of transformation (Fig. 13). Attracting operations in parametric design act like magnets that affect the objects. By combining the attraction method and parquet deformation, we shall consider one of the pattern sides as an attractor and finally, we can parametrize this method in the Grasshopper plugin. Because attractors can adjust the position, rotation or scale of the patterns, they can create more designs in the way of developing the deformation method (Fig. 14).

We can use this method in space tessellation as well, by modifying the patterns in the z-axis. If the repeat unit (the cube, for example) moves toward one axis, the results are similar to plane patterns. However, in 3D transformation, the result is different and employs a parametric method.

Architecture and Design

Islamic geometric patterns had a great influence in architectural forms, especially façades and ornaments. Islamic decoration and architecture are bound together and can't be separated from each other. That means that in developing Islamic ornaments,

we should also consider the application of this design in architecture. The Masdar City project in Abu Dhabi is an example of the projects representing traditional design alongside modern tools such as sustainability and zero-energy buildings. For the balconies of the residential buildings, they used Islamic patterns, under the supervision of Jean-Marc Castera. The result was a contemporary reinterpretation of the mashrabiya screen, which protected the second inner layer, including windows (Palmer 2011).

Here we do not review or criticize the project elements but create some alternate examples for the design with the same ideology. We are showing samples that follow the project concepts, which are sustainable developments alongside establishing the traditional design. As we introduced the attracting operation in the previous part, our proposed design can attach to the sunlight path as a parameter and can respond to the sky lighting in real-time to control the lighting of the building (Fig. 15a–c). The idea of developing the traditional planar patterns retains the 2D specifications in the façade of the building and also can associate with contemporary needs, design, growth and facilities.

4D Space

Now we will briefly discuss the geometrical transformation of the patterns into the next dimensions, a 4D Islamic geometric space pattern here. 4D space is a mathematical abstract and imaginary concept stemming from the generalization of the law of 3D space. Philosophers and mathematicians have studied this topic for almost three hundred years. The ideology and geometric concepts behind four dimensions are complicated, and opening this topic is very detailed; however, Florian Cajori (1926) described the origin and concept very well. The human cannot perceive four dimensions spatially and for that reason, we are using the shadows of 4D that are projected into lower dimensions for understanding the situation of forms in space. The process of geometric transformation is the same as that shown in Table 2, which starts with the repeat unit. The cube in the 4D space is called a tesseract, octachoron, octahedron, or eight-cell, which is a hypercube in $\mathbb{R}^4$.

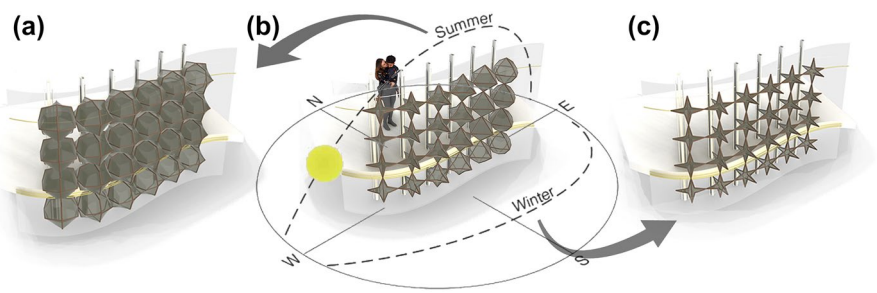


Fig. 15 A proposed façade for balcony in residential buildings; **a** 3D Islamic geometric pattern modules with full close mode, in summer **b** the modules that track sunlight as the parameter for controlling the lighting; **c** the modules with full open mode, in winter

The tesseract can be constructed in several ways. For example, with hyperoctahedral symmetry, it can be made with three cubes folded together. The hypercube made up of eight hyperplanes that consist of eight cubes, twenty-four squares, thirty-two edges, and sixteen vertices. These eight cubes can form the Islamic geometric pattern repeat unit induvial, as shown in Table 2. Finally, by constructing the tesseract, using transformed cubes, we can produce a 4D Islamic geometric pattern (Fig. 16, GIF 1 and 2). The usage of this dimension projecting is more a fundamental knowledge rather than applied design and art. We can mention here that the 4D space pattern defaults to a demonstration and employment as an ornament or art design. The purpose of this section was to show the capability of this method for constructing patterns in higher dimensions rather than the application of them.

Conclusion

In the construction section, we described methods for creating 3D Islamic geometric patterns. We gave a brief introduction to modern mathematics that we required for our method to produce periodic and aperiodic tessellation with our repeat units. We showed that it is possible to construct 3D Islamic geometric patterns with a deductive approach, a geometric transformation from 2D design to 3D that employ historical (studied) methods, and an inductive approach, which uses operations such as truncation to produce new solids and higher dimensions like tesseract Islamic geometric patterns.

Although there is no evidence that Islamic artists used matching rules for creating quasiperiodic patterns (Cromwell 2009), here we introduced an Islamic geometric pattern construction for creating aperiodic tilings in three dimensions that can fill the space. We have not discussed Muslims' awareness of quasiperiodic order but demonstrated that how we can make an effort on their works by advancing their way of thinking and ideas.

Islamic geometric patterns in three dimensions have different behavior than their planar origins. The first challenging difference is the coloring process of the patterns in 3D shapes. Most of the historical examples are colored regions

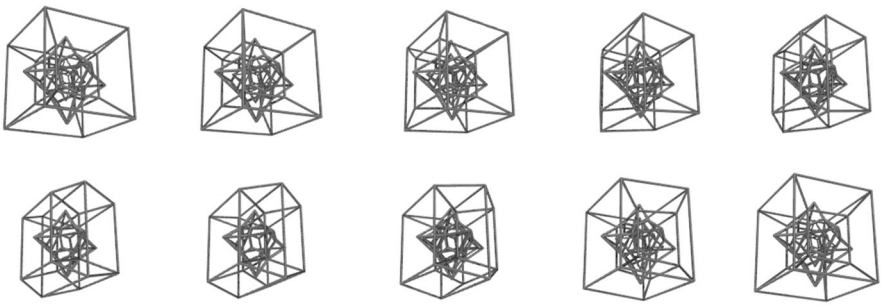


Fig. 16 The 4D (tesseract) repeat unit of Islamic geometric patterns in different captured shadows

Table 4 Semi-regular Islamic geometric patterns tessellation: cubes, truncated octahedra, and truncated cuboctahedra

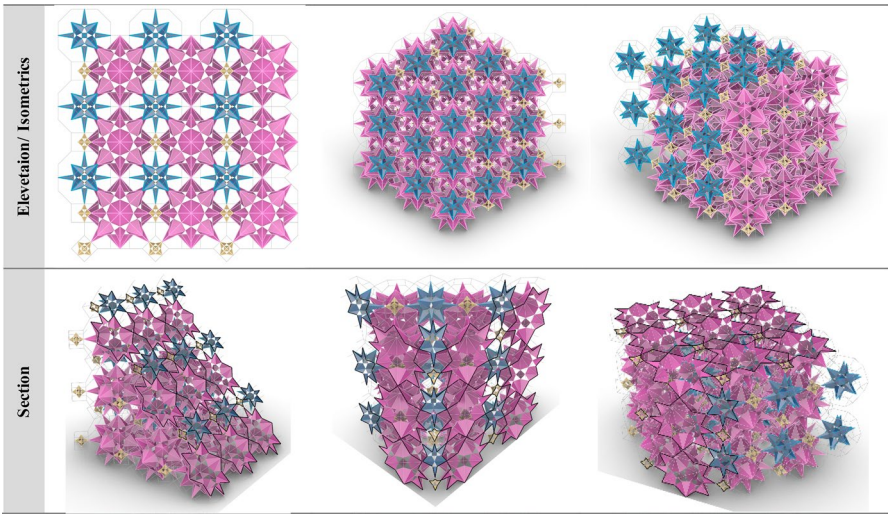
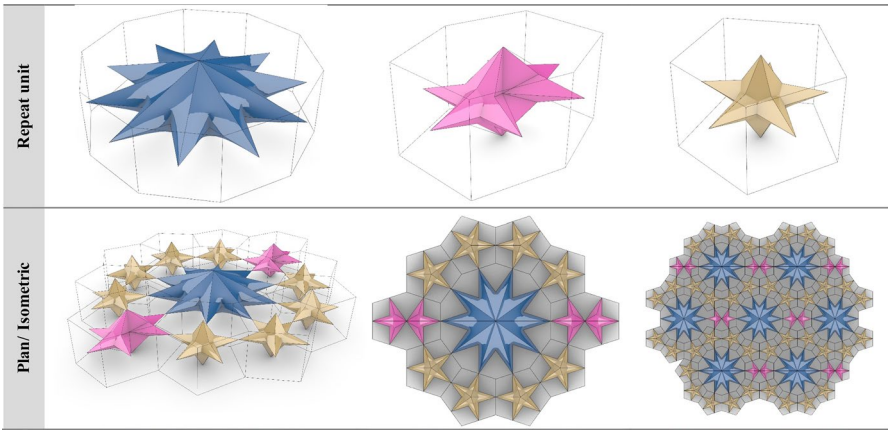


Table 5 Regular and irregular Islamic geometric pattern repeat units and their tessellation: ten-pointed star, barrel-shaped hexagonal and five-pointed star



bounded by interlaced strands (Kaplan and Salesin 2004); therefore coloring the pattern in a higher dimension is a decorated and symmetrical stage as well as a productive one. Grünbaum and Shephard (1987) discussed the concept of "colored patterns and tilings" and described how monochrome or chromatic coloring can create variant symmetry in the tiles. In our result, we showed the foreground with the blue color and left the background transparent as the monochrome tiling. However, this is the simplest method for coloring and generating patterns. By coloring primitive patterns in chromatic or multi-color (Tables 4 and 5), we can create the same designs but with different tessellation

symmetry. These new coloring patterns generate a whole new color group model that can proceed in a separate study.

The final goal in our process of study is to employ the outputs in contemporary utilization for the continuation of the Muslim heritage. We believe that the application of historical patterns and ornaments alongside newly developed patterns can create suitable spaces in Islamic contents. Conceptual designs, products, and buildings are appropriate targets for our proposed method and results. To us, it seems most likely that the Islamic artists and mathematicians were interested in advancing their knowledge and arts. All that we have done was an extension of their knowledge by using modern mathematics and geometry to pursue our goal.

Acknowledgements All the graphical illustrations are done by the authors.

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Sam Moradzadeh is an architect and an academic researcher in the area of computational design and interactive architecture. His current master's thesis focuses on the expansion of Islamic geometric patterns in three dimensions in Islamic architecture at Tabriz Islamic Art University. He is interested in the intersection of geometry, art, and mathematics, everything with a strong visual element. He is working on prototyping and fabricating creative shapes and geometric patterns using CAD/CAM.

Ahad Nejad Ebrahimi is a practicing architect in Iran and an associate professor at the faculty of Architecture and Urbanism, Tabriz Islamic Art University, Iran. He holds a PhD degree in Architecture. During his fully-funded study at the Faculty of Art and Architecture, Tarbiat Modares University of Tehran, he managed to publish several research works. His current research and teaching areas are as follows: Architectural Theory, Architectural Design, Urban Conservation, and Islamic Architecture. Ahad always made a great effort trying to support national heritage and conservation of architecture and urbanism. He also actively contributed to the development of a program for conservation and restoration of historic buildings and urban areas in Iran.